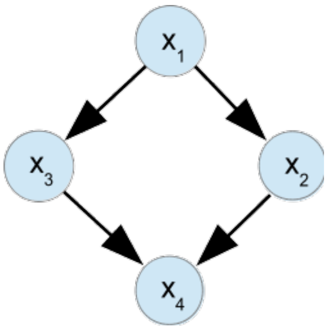
MACHINE LEARNING 1, WS2019/20
2. EXERCISE

- proofs, sat, etc -

problem 1. The joint probability in the variables $x1,\ldots,x7$ shall be given as $p(x1,x2,x3,x4,x5,x6,x7) = p(x1)p(x3)p(x7)p(x2|x1,x3)p(x5|x7,x2)p(x4|x2)p(x6|x5,x4)$. Draw the directed graphical model for this joint probability!

problem 2. Write down the joint probability for the DGM given in the pictures:



problem 3. Logistic regression (Exercise from the lecture notes):
We consider

- The logistic distribution

$$\forall s \in S: \qquad p_{Y_s|X_s,\Theta}(1) = \frac{1}{1+2^{-\langle\theta,x_s\rangle}} \tag{1}$$

- A $\sigma \in \mathbb{R}^+$ and the normal distribution:

$$\forall v \in V: \qquad p_{\Theta_v}(\theta_v) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\theta_v^2/2\sigma^2} \tag{2}$$

Estimating maximally probable parameters θ, given features x and labels y, means solving the optimization problem

$$\begin{aligned}
&\underset{\theta\in\mathbb{R}^m}{\operatorname{argmax}} \quad p_{\Theta|X,Y}(\theta,x,y) \\
&= \underset{\theta\in\mathbb{R}^m}{\operatorname{argmax}} \quad \prod_{s\in S} p_{Y_s|X_s,\Theta}(y_s,x_s,\theta) \prod_{v\in V} p_{\Theta_v}(\theta_v) \\
&= \underset{\theta\in\mathbb{R}^m}{\operatorname{argmax}} \quad \sum_{s\in S} \log p_{Y_s|X_s,\Theta}(y_s,x_s,\theta) + \sum_{v\in V} \log p_{\Theta_v}(\theta_v)
\end{aligned} \tag{3}$$

Substituting in (3) the linearization

$$\begin{aligned} & \log p_{Y_s|X_s,\Theta}(y_s, x_s, \theta) \\ = \quad & y_s \log p_{Y_s|X_s,\Theta}(1, x_s, \theta) + (1 - y_s) \log p_{Y_s|X_s,\Theta}(0, x_s, \theta) \\ = \quad & y_s \log \frac{p_{Y_s|X_s,\Theta}(1, x_s, \theta)}{p_{Y_s|X_s,\Theta}(0, x_s, \theta)} + \log p_{Y_s|X_s,\Theta}(0, x_s, \theta) \end{aligned} \tag{4}$$

as well as (1) and (2) yields the form (5) below that is called the instance of the l_2-regularized logistic regression problem with respect to x, y and σ.

$$\underset{\theta \in \mathbb{R}^m}{\operatorname{argmin}} \quad \sum_{s \in S} \left(-y_s \langle \theta, x_s \rangle + \log \left(1 + 2^{\langle \theta, x_s \rangle} \right) \right) + \frac{\log e}{2\sigma^2} \|\theta\|_2^2 \tag{5}$$

Exercise (Logistic regression):

a) Derive (5) from (3) using (4), (1) and (2)
b) Is the objective function of (5) convex?

problem 4. Follow the proof of Theorem 3-SAT $\leq_p$ 3-PM in order to:

a) construct the instance of 3-PM for the instance of 3-SAT given by the 3-CNF $(x_1 \vee (1 - x_2) \vee x_3) \cdot ((1 - x_1) \vee x_2 \vee x_4)$.
b) construct, for any solution to this instance of 3-SAT, the solution to the instance of 3-PM.

problem 5. Complete the proof for COLORING is NP-complete sketched in the script by showing that the instance of 3-SAT has a solution iff (!) G is 3-colorable.
Hint: Show that f_χ necessarily has the same color as 1. Examine the implication of f_χ having the same color as 1 on the feasible colorings of c_χ and l_3.