6.3.7 Inference algoritms

Iterated conditional modes (ICM)

For the inference problem

$$\underset{y \in \{0,1\}^S}{\operatorname{argmin}} \; H_\theta(x, y) \quad , \tag{6.41}$$

a heuristic that is guaranteed to converge and terminate in a (possibly sub-optimal) feasible solution is local search w.r.t. transformations that change one variable at a time:

Definition 14 For any $s \in S$, let $\operatorname{flip}_s \colon \{0,1\}^S \to \{0,1\}^S$ such that for any $y \in \{0,1\}^S$ and any $t \in S$:

$$\operatorname{flip}_s[y](t) = \begin{cases} 1 - y_t & \text{if } t = s \\ y_t & \text{otherwise} \end{cases} \quad . \tag{6.42}$$

Algorithm 2 Greedy local search w.r.t. transformations that change one variable at a time is defined by the recursion below. In the context of graphical models and probabilistic inference, this algorithm is also known as *iterated conditional modes*, or ICM (**?**).

$y' = \operatorname{icm}(y)$
choose $s \in \underset{s' \in S}{\operatorname{argmin}} \; H_\theta(x, \operatorname{flip}_{s'}[y]) - H_\theta(x, y)$
if $H_\theta(x, \operatorname{flip}_s[y]) < H_\theta(x, y)$
$\quad y' := \operatorname{icm}(\operatorname{flip}_s[y])$
else
$\quad y' := y$

Message passing

The inference problem

$$\underset{y \in \{0,1\}^S}{\operatorname{argmin}} \sum_{f \in F} h_{f\theta}(x_f, y_{S(f)}) \tag{6.43}$$

consists in computing the minimum of a sum of of functions. This problem is analogous to that of computing the sum of a product of functions (Section 6.3.6) in that both $(\mathbb{R}, \min, +)$ and $(\mathbb{R}, +, \cdot)$ are commutative semi-rings. This analogy is sufficient to transfer the idea of message passing, albeit with messages adapted to the $(\mathbb{R}, \min, +)$ semi-ring:

Definition 15 (Kschischang et al. (2001)) For any variable node $s \in S$ and any factor node $f \in F$, the functions

$$\mu_{s \to f}, \mu_{f \to s} \colon \{0,1\} \to \mathbb{R} \quad , \tag{6.44}$$

called *messages*, are defined such that for all $y_s \in \{0,1\}$:

$$\mu_{s \to f}(y_s) = \sum_{f' \in F(s) \setminus \{f\}} \mu_{f' \to s}(y_s) \tag{6.45}$$

$$\mu_{f \to s}(y_s) = \min_{y_{S(f) \setminus \{s\}}} \psi_{f\theta}(x_f, y_{S(f)}) + \sum_{s' \in S(f) \setminus \{s\}} \mu_{s' \to f}(y_{s'}) \tag{6.46}$$

Lemma 11 *If the factor graph is acyclic, messages are defined recursively by* (6.45) *and* (6.46)*, beginning with the messages from leaves. Moreover, for any $s \in S$:*

$$\min_{y \in \{0,1\}^S} \sum_{f \in F} h_{f\theta}(x_f, y_{S(f)}) = \min_{y_s \in \{0,1\}} \sum_{f' \in F(s)} \mu_{f' \to s}(y_s)$$

PROOF Analogous to Lemma 10.

For conditional graphical models whose factor graph is acyclic, the inference problem can be solved efficiently by means of message passing, by Lemma 11.

For conditional graphical models whose factor graph is cyclic, the definition of messages by (6.45) and (6.46) is cyclic as well. The inference problem cannot be solved by message passing in general. A heuristic without guarantee of correctness or even convergence is to initialize all messages as constant zero and update messages according to some schedule, e.g., synchronously. This heuristic is also known as *loopy belief propagation* and has proven suitable for some applications.