

Machine Learning II

Bjoern Andres

Machine Learning for Computer Vision
TU Dresden



<https://mlcv.cs.tu-dresden.de/courses/26-summer/ml2/>

Summer Term 2026

Ordering

Definition 1. A strict order on a set A is a relation $< \subseteq A \times A$ such that:

$$\forall a \in A: \quad \neg a < a \quad (1)$$

$$\forall a \in A \forall b \in A \setminus \{a\}: \quad a < b \text{ xor } b < a \quad (2)$$

$$\forall a \in A \forall b \in A \setminus \{a\} \forall c \in A \setminus \{a, b\}: \quad a < b \wedge b < c \Rightarrow a < c \quad (3)$$

Lemma 1. The strict orders on A are characterized by the bijections $\alpha : \{0, \dots, |A| - 1\} \rightarrow A$.

Proof (sketch). For any such bijection, consider the relation $<_{\alpha}$ such that

$$\forall a, b \in A: \quad a < b \Leftrightarrow \alpha^{-1}(a) < \alpha^{-1}(b) . \quad (4)$$

Notation: For any set A , let $S_A := \{ab \in A^2 \mid a \neq b\}$.

Lemma 2. The strict orders on A are characterized by those $y \in \{0, 1\}^{S_A}$ that satisfy the following conditions:

$$\forall a \in A \forall b \in A \setminus \{a\}: \quad y_{ab} + y_{ba} = 1 \quad (5)$$

$$\forall a \in A \forall b \in A \setminus \{a\} \forall c \in A \setminus \{a, b\}: \quad y_{ab} + y_{bc} - 1 \leq y_{ac} \quad (6)$$

Ordering

Definition 2 (Grötschel, 1984; see also Martí and Reinelt, 2011). An instance of the **minimum cost linear ordering problem** is a pair (A, c) with A a finite set and $c \in \mathbb{R}^{S_A}$. Its feasible set is

$$Y_A := \{y \in \{0, 1\}^{S_A} \mid (5) \text{ and } (6)\} . \quad (7)$$

Its solutions are those $y^* \in Y_A$ with

$$\langle c, y^* \rangle = \min\{\langle c, y \rangle \mid y \in Y_A\} . \quad (8)$$

An instance of the **linear ordering problem** is a triple (A, c, k) with A a finite set, $c \in \mathbb{R}^{S_A}$ and $k \in \mathbb{R}$. Its solution set is

$$\text{LOP}(A, c, k) = \{y \in Y_A \mid \langle c, y \rangle \leq k\} . \quad (9)$$

Theorem 1. The linear ordering decision problem is NP-complete.

It is obviously in NP. A well-known proof of its NP-hardness is by reduction of the feedback arc set problem:

Definition 3. For any digraph $G = (V, E)$ and any $F \subseteq E$, the set F is called a **feedback arc set** of G iff the graph $(V, E \setminus F)$ is acyclic.

Definition 4. An instance of the **feedback arc set problem** is a triple (V, E, m) such that $G = (V, E)$ is a digraph and $m \in \mathbb{N}_0$. Its solution set is

$$\text{FAS}(V, E, m) = \{F \subseteq E \mid F \text{ is a feedback arc set of } (V, E) \text{ and } |F| \leq m\} \quad (10)$$

Theorem 2 (Karp 1972). The feedback arc set problem is NP-complete.

Ordering: Hardness

Proof of Theorem 1 (sketch): Given an instance (V, E, m) of the feedback arc set problem, consider the instance (V, c, k) of the linear ordering problem with $k := m - |E|$ and $c \in \mathbb{R}^S$ such that

$$\forall ab \in S_V: \quad c_{ab} := \begin{cases} -1 & \text{if } ab \in E \\ 0 & \text{otherwise} \end{cases} . \quad (11)$$

For any $y \in \{0, 1\}^{S_V}$:

$$E = (E \cap y^{-1}(0)) \dot{\cup} (E \cap y^{-1}(1)) . \quad (12)$$

(\Rightarrow) Let $y \in \text{LOP}(V, c, k)$ arbitrary. $y^{-1}(1)$ is acyclic. Therefore, $y^{-1}(1) \cap E$ is acyclic. Thus, by (12), $E \setminus (y^{-1}(0) \cap E)$ is acyclic. Furthermore,

$$|E \cap y^{-1}(0)| \stackrel{(12)}{=} |E| - |E \cap y^{-1}(1)| = |E| + \langle c, y \rangle \leq |E| + (m - |E|) = m$$

Thus, $(y^{-1}(0) \cap E) \in \text{FAS}(V, E, m)$.

(\Rightarrow) Let $F \in \text{FAS}(V, E, m)$ arbitrary. There exists a linear order and corresponding y such that $(E \cap y^{-1}(1)) = E \setminus F$. Therefore,

$$\langle c, y \rangle = -|E \cap y^{-1}(1)| = -|E \setminus F| = -|E| + |F| \leq -|E| + m = k . \quad (13)$$

Thus, $y \in \text{LOP}(V, c, k)$. □

Notation:

- For any $r \in \mathbb{R}$, let $r^\pm := \max\{0, \pm r\}$.
- For any instance (A, c) of the linear ordering problem and any $ab \in S_A$, let $c_{ab}^\Delta := c_{ab} - c_{ba}$.

Theorem 3. For any instance (A, c) of the linear ordering problem and any $ab \in S_A$ such that

$$c_{ab}^\Delta + \sum_{d \in A \setminus \{a, b\}} (c_{ad}^\Delta + c_{db}^\Delta)^+ \leq 0, \quad (14)$$

there exists an optimal solution y^* to this instance such that $y_{ab}^* = 1$.

Ordering: Partial optimality

We prove Theorem 3 by showing that, under the premise of the theorem, the map defined below is improving for the instance (A, c) of the linear ordering problem.

Definition 5. For any finite set A and any $\{a, b\} \in \binom{A}{2}$, define **transposition** $\tau_{\{a,b\}}: Y_A \rightarrow Y_A$ such that for all $y \in Y_A$ and all $a'b' \in S_A$:

$$\tau_{\{a,b\}}(y)_{a'b'} = \begin{cases} 1 - y_{a'b'} & \text{if } \{a', b'\} = \{a, b\} \\ y_{a'a} & \text{if } b' = b \wedge a' \neq a \\ y_{ab'} & \text{if } a' = b \wedge b' \neq a \\ y_{bb'} & \text{if } a' = a \wedge b' \neq b \\ y_{a'b} & \text{if } b' = a \wedge a' \neq b \\ y_{a'b'} & \text{if } \{a, b\} \cap \{a', b'\} = \emptyset \end{cases} . \quad (15)$$

Define **conditional transposition** $\tau_{\{a,b\}}^1: Y_A \rightarrow Y_A$ such that for all $y \in Y_A$:

$$\tau_{ab}^1(y) = \begin{cases} x & \text{if } y_{ab} = 1 \\ \tau_{\{a,b\}}(y) & \text{otherwise} \end{cases} . \quad (16)$$

To begin with, we calculate the effect of transposition on the objective value:

Lemma 3. Let (A, c) be any instance of the linear ordering problem, let $y \in Y_A$, let $\{a, b\} \in \binom{A}{2}$, and let $y' = \tau_{\{a,b\}}(y)$. Then:

$$\langle c, y' \rangle - \langle c, y \rangle = c_{ab}^{\Delta}(1 - 2y_{ab}) + \sum_{d \in A \setminus \{a,b\}} (c_{ad}^{\Delta} + c_{db}^{\Delta})(y_{bd} + y_{da} - 1) . \quad (17)$$

Ordering: Partial optimality

Proof.

$$\begin{aligned}
 \langle c, y' \rangle - \langle c, y \rangle &= \sum_{a'b' \in S_A} c_{a'b'}(y'_{a'b'} - y_{a'b'}) = \sum_{\substack{a'b' \in S_A \\ \{a', b'\} \cap \{a, b\} \neq \emptyset}} c_{a'b'}(y'_{a'b'} - y_{a'b'}) \\
 &= c_{ab}(y'_{ab} - y_{ab}) + c_{ba}(y'_{ba} - y_{ba}) \\
 &\quad + \sum_{b' \in A \setminus \{a, b\}} c_{ab'}(y'_{ab'} - y_{ab'}) + \sum_{a' \in A \setminus \{a, b\}} c_{a'a}(y'_{a'a} - y_{a'a}) \\
 &\quad + \sum_{b' \in A \setminus \{a, b\}} c_{bb'}(y'_{bb'} - y_{bb'}) + \sum_{a' \in A \setminus \{a, b\}} c_{a'b}(y'_{a'b} - y_{a'b}) \\
 &= c_{ab}(1 - 2y_{ab}) + c_{ba}(1 - 2y_{ba}) \\
 &\quad + \sum_{d \in A \setminus \{a, b\}} c_{ad}(y_{bd} - y_{ad}) + \sum_{d \in A \setminus \{a, b\}} c_{da}(y_{db} - y_{da}) \\
 &\quad + \sum_{d \in A \setminus \{a, b\}} c_{bd}(y_{ad} - y_{bd}) + \sum_{d \in A \setminus \{a, b\}} c_{db}(y_{da} - y_{db}) \\
 &= (c_{ab} - c_{ba})(1 - 2y_{ab}) + \sum_{d \in A \setminus \{a, b\}} (y_{da} + y_{bd} - 1)(c_{ad} - c_{da} - c_{bd} + c_{db}) \\
 &= c_{ab}^{\Delta}(1 - 2y_{ab}) + \sum_{d \in A \setminus \{a, b\}} (y_{da} + y_{bd} - 1)(c_{ad}^{\Delta} + c_{db}^{\Delta})
 \end{aligned}$$

□

Ordering: Partial optimality

Next, we establish a basic property of linear orders:

Lemma 4. For any finite set A , any $y \in Y_A$ such that $y_{ab} = 1$, and any $c \in A \setminus \{a, b\}$:

$$1 \leq y_{ac} + y_{cb} \tag{18}$$

Proof. For any $c \in A \setminus \{a, b\}$, we have

$$x_{bc} + x_{ca} - 1 \leq x_{ba} .$$

On the left-hand side,

$$x_{bc} + x_{ca} - 1 = (1 - x_{cb}) + (1 - x_{ac}) - 1 = 1 - x_{ac} - x_{cb} .$$

On the right-hand-side,

$$x_{ba} = 1 - x_{ab} = 1 - 1 = 0 .$$

Thus:

$$1 \leq x_{ac} + x_{cb} .$$

□

Ordering: Partial optimality

Proof of Theorem 3. Let y be any solution to the instance of the linear ordering problem, and let $y' = \tau_{ab}^1(y)$. If $y_{ab} = 1$ then $y' = y$. Thus, $\langle c, y' \rangle = \langle c, y \rangle$. If $y_{ab} = 0$ then

$$\begin{aligned}
 & \langle c, y' \rangle - \langle c, y \rangle \\
 &= c_{ab}^\Delta + \sum_{d \in A \setminus \{a, b\}} (x_{da} + x_{bd} - 1)(c_{ad}^\Delta + c_{db}^\Delta) && \text{by Lemma 3} \\
 &\leq c_{ab}^\Delta + \max_{\substack{x \in X_A \\ x_{ab} = 0}} \sum_{d \in A \setminus \{a, b\}} (x_{da} + x_{bd} - 1)(c_{ad}^\Delta + c_{db}^\Delta) \\
 &\leq c_{ab}^\Delta + \max_{\substack{x \in X_A \\ x_{ab} = 0}} \sum_{d \in A \setminus \{a, b\}} (x_{da} + x_{bd} - 1)(c_{ad}^\Delta + c_{db}^\Delta)^+ && \text{by Lemma 4} \\
 &= c_{ab}^\Delta + \sum_{d \in A \setminus \{a, b\}} (c_{ad}^\Delta + c_{db}^\Delta)^+ && \text{as } x_{da} + x_{bd} - 1 \leq 1 \\
 &\leq 0 && \text{by the premise .}
 \end{aligned}$$

Therefore, τ_{ab}^1 is improving. Thus, y' is a solution to the instance (A, c) of the linear ordering problem. Moreover, $y'_{ab} = 1$. □