Computer Vision I

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Machine Learning for Computer Vision
TU Dresden

Winter Term 2022/2023

Motivation

- Many geometric calculations in the field of computer vision have a simpler algebraic form in the coordinates of a projective space than in the coordinates of a vector space, or require less case distinctions.
- APIs of computer vision software as well as GPU hardware are designed for these forms.

Literature

- Hartley, R. I. and Zisserman, A.. Multiple View Geometry in Computer Vision. Second edition. 2004. Cambridge University Press

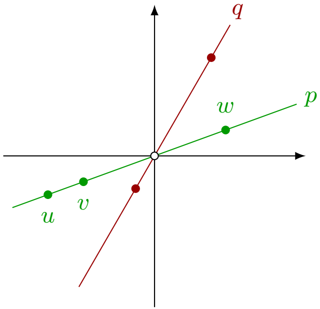
Definition. For any vector space V over a field K, the **same bi-ray relation** is the binary relation $\sim$ over $V \setminus \{0\}$ such that for all $v, w \in V \setminus \{0\}$, we have

$$v \sim w \quad \Leftrightarrow \quad \exists k \in K \setminus \{0\}\colon \; w = k\,v \tag{1}$$

Lemma. The same bi-ray relation is an equivalence relation.

Definition. The equivalence classes of the same ray relation are called **bi-rays**.

Example. $V = \mathbb{R}^2$



Definition. For any vector space V, the **projective space** $P(V)$ is the set of **bi-rays**.

In case $V = K^{n+1}$ for some $n \in \mathbb{N}$, we write $P_n(K)$ instead of $P(V)$ and call it the[1] n**-dimensional projective space**.

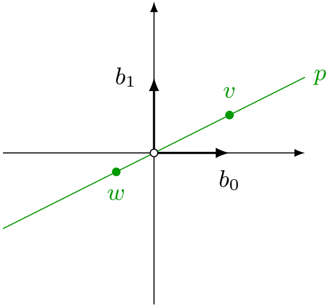
$P_n(\mathbb{R})$	n-dimensional **real projective space**
$P_2(\mathbb{R})$	**projective plane**
$P_1(\mathbb{R})$	**projective line**

[1] Recall: Every K vector space of dimension $n + 1$ is isomorphic to K^{n+1}.

Definition. Fix a K vector space V and a basis B of V. For any $p \in P(V)$ and any $c\colon B \to K$ such that $\sum_{b\in B} c_b b \in p$, the coordinates c are called **projective coordinates** of p.

Lemma. Fix a K vector space V and a basis B of V. For any projective coordinates c, c' of the same point p, there exists a $\lambda \in K \setminus \{0\}$ such that $c' = \lambda c$.

Example. $P_1(\mathbb{R})$



Notation:

$$p = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{4} \end{bmatrix}$$

Square brackets indicate equivalence classes

For a function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, the condition $f(v) = 0$ does not necessarily well-define a subset of $P_n(\mathbb{R})$.

Example: Consider $f(v) = v_0 v_1 + v_0$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ and observe that $f(\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]) = 0 \neq -2 = f(\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right])$.

Definition. A function $f\colon V \to W$ between two K vector spaces is called **homogenous** of degree $k \in \mathbb{N}$ if

$$\forall v \in V \ \forall \lambda \in K \setminus \{0\}\colon \quad f(\lambda\, v) = \lambda^k f(v) \tag{2}$$

Lemma. For any homogenous function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$, the set $\{v \in P_n(\mathbb{R}) \mid f(v) = 0\}$ is well-defined.

Lemma. For any polynomial function $f\colon \mathbb{R}^n \to \mathbb{R}$ of degree k, the function $g\colon \mathbb{R}^{n+1} \to \mathbb{R}\colon v \mapsto v_n^k\, f(v/v_n)$ is a homogenous polynomial function of degree k. Moreover:

$$\forall v \in \mathbb{R}^n\colon \quad f(v_0, \ldots, v_{n-1}) = g(v_0, \ldots, v_{n-1}, 1) \tag{3}$$

Let $n \in \mathbb{N}$ and $n \geq 2$.

Any **point** $v \in \mathbb{R}^n$ may be represented as the bi-ray $[v_0, \ldots, v_{n-1}, 1] \in P_n(\mathbb{R})$.

Bi-rays $[w_0, \ldots, w_{n-1}, 0] \in P_n(\mathbb{R})$ do not represent points in $\mathbb{R}^2$.

We may augment $\mathbb{R}^n$ by **points at infinity** $\infty(w_0, \ldots, w_{n-1})$ represented by these bi-rays.

Any $(n-1)$-**dimensional hyperplane** $\{v \in \mathbb{R}^n \mid c_0 v_0 + \cdots + c_{n-1} v_{n-1} + c_n = 0\}$ may be represented as the bi-ray $c := [c_0, \ldots, c_n] \in P_n(\mathbb{R})$.

We have $(c_0, \ldots, c_{n-1}) \neq 0$. (Otherwise, the hyperplane would not have dimension $n-1$). Thus, the bi-ray $[0, \ldots, 0, 1] \in P_n(\mathbb{R})$ does not represent an $(n-1)$-dimensional hyperplane.

We may associate with it an $(n-1)$-**dimensional hyperplane at infinity**.

Lemma. A point $v \in P_n(\mathbb{R})$ lies on an $(n-1)$-dimensional hyperplane $c \in P_n(\mathbb{R})$ if and only if $c^T v = 0$. (Precisely the points at infinity lie on the $(n-1)$-dimensional hyperplane at infinity).